M4 – FURTHER MOTION OF PARTICLES IN ONE DIMENSION

Question:

A train, total mass M, including the engine, is moving along a straight horizontal track. The engine exerts a constant driving force of magnitude F. At any instant the total resistance is bv^2 where b is a positive constant and v is the speed of the train at that instant. Show that the limiting speed V of the train is $\left(\frac{F}{b}\right)^{\frac{1}{2}}$.

The train starts from rest at $t = 0$. Show that it reaches a speed of $\frac{1}{2}V$ after a time

$$\frac{M}{2bV}\ln 3$$

Show further that the distance covered in this time is

$$\frac{M}{2b}\ln\left(\frac{4}{3}\right)$$

[*Edexcel*]

Solution:

The limiting speed V is reached when the resultant force acting on the train is zero. That is:

$$F - bV^2 = 0 \quad \Rightarrow \quad F = bV^2 \quad \Rightarrow \quad V^2 = \frac{F}{b} \quad \Rightarrow \quad V = \left(\frac{F}{b}\right)^{\frac{1}{2}}$$

For the next part:

$$M\frac{\mathrm{d}v}{\mathrm{d}t} = F - bv^2 \quad \Rightarrow \quad \frac{\mathrm{d}v}{\mathrm{d}t} = \frac{F - bv^2}{M}$$

$$\int \frac{\mathrm{d}v}{F - bv^2} = \int \frac{1}{M}\,\mathrm{d}t$$

$$\frac{1}{b}\int \frac{\mathrm{d}v}{\left(\frac{F}{b}\right) - v^2} = \int \frac{1}{M}\,\mathrm{d}t$$

$$\frac{1}{2b\sqrt{\frac{F}{b}}}\int \left(\frac{1}{\sqrt{\frac{F}{b}} + v} + \frac{1}{\sqrt{\frac{F}{b}} - v}\right)\mathrm{d}v = \frac{t}{M} + C$$

$$\frac{1}{2b\sqrt{\frac{F}{b}}}\left[\ln\left(\sqrt{\frac{F}{b}} + v\right) - \ln\left(\sqrt{\frac{F}{b}} - v\right)\right] = \frac{1}{2b\sqrt{\frac{F}{b}}}\ln\left(\frac{\sqrt{\frac{F}{b}} + v}{\sqrt{\frac{F}{b}} - v}\right) = \frac{t}{M} + C$$

But when $t = 0$, $v = 0$, therefore $C = 0$. And since $\left(\frac{F}{b}\right)^{\frac{1}{2}} = V$,

$$\therefore \quad t = \frac{M}{2bV}\ln\left(\frac{V+v}{V-v}\right)$$

When $v = \tfrac{1}{2}V$,

$$t = \frac{M}{2bV}\ln\left(\frac{V+\tfrac{1}{2}V}{V-\tfrac{1}{2}V}\right) = \frac{M}{2bV}\ln\left(\frac{\tfrac{3}{2}V}{\tfrac{1}{2}V}\right)$$

$$\therefore \quad t = \frac{M}{2bV}\ln 3$$

For the last part:

$$v\frac{\mathrm{d}v}{\mathrm{d}x} = \frac{F-bv^2}{M}$$

$$\int \frac{v\,\mathrm{d}v}{F-bv^2} = \int \frac{1}{M}\,\mathrm{d}x$$

$$-\frac{1}{2b}\ln\left(F-bv^2\right) = \frac{x}{M} + A$$

Since $v = 0$ when $x = 0$, $A = -\frac{1}{2b}\ln F$

$$\frac{x}{M} = \frac{1}{2b}\ln F - \frac{1}{2b}\ln\left(F-bv^2\right)$$

$$\therefore \quad x = \frac{M}{2b}\ln\left(\frac{F}{F-bv^2}\right)$$

Since $V = \left(\frac{F}{b}\right)^{\frac{1}{2}} \quad \Rightarrow \quad V^2 = \frac{F}{b} \quad \Rightarrow \quad F = bV^2$

$$\therefore \quad x = \frac{M}{2b}\ln\left(\frac{bV^2}{bV^2 - bv^2}\right)$$

$$v = \tfrac{1}{2}V \quad \Rightarrow \quad v^2 = \tfrac{1}{4}V^2$$

$$x = \frac{M}{2b}\ln\left(\frac{bV^2}{\tfrac{3}{4}bV^2}\right)$$

$$\therefore \quad x = \frac{M}{2b}\ln\left(\frac{4}{3}\right)$$