P5 – CALCULUS

Question:

(a) Given that $u = \frac{1}{2}\left(e^{y} - e^{-y}\right)$, prove that $y = \ln\left(u + \sqrt{u^2+1}\right)$.

(b) Using the substitution $x = \sinh u$, show that

$$\int \frac{x^2}{\sqrt{1+x^2}}\,dx = \frac{1}{2}\left[x\sqrt{1+x^2} - \ln\left(x + \sqrt{1+x^2}\right)\right] + k$$

where k is an arbitrary constant.

[*Edexcel*]

Solution:

(a)

$$u = \frac{1}{2}\left(e^{y} - e^{-y}\right)$$
$$2u = e^{y} - e^{-y}$$
$$e^{2y} - 2ue^{y} - 1 = 0$$
$$e^{y} = \frac{2u \pm \sqrt{4u^2+4}}{2} = \frac{2u \pm 2\sqrt{u^2+1}}{2} = u \pm \sqrt{u^2+1}$$

Since $\sqrt{u^2+1} > u$ and $e^{y} > 0$

$$e^{y} = u + \sqrt{u^2+1}$$
$$\underline{\underline{y = \ln\left(u + \sqrt{u^2+1}\right)}}$$

(b)

$$x = \sinh u \quad \to \quad \frac{dx}{du} = \cosh u$$

$$\int \frac{x^2}{\sqrt{1+x^2}}\,dx = \int \frac{\sinh^2 u}{\sqrt{1+\sinh^2 u}} \cdot \cosh u\,du = \frac{\sinh^2 u}{\sqrt{\cosh^2 u}} \cdot \cosh u\,du = \int \frac{\sinh^2 u}{\cosh u} \cdot \cosh u\,du$$
$$= \int \sinh^2 u\,du$$

Now by the hyperbolic double-angle formulae:

$$\int \sinh^2 u\,du = \frac{1}{2}\int\left(\cosh 2u - 1\right)du$$
$$= \frac{1}{2}\left[\frac{1}{2}\sinh(2u) - u\right] + k$$

But from (a): $x = \sinh u = \frac{1}{2}\left(e^{u} - e^{-u}\right) \quad \Rightarrow \quad u = \ln\left(x + \sqrt{x^2+1}\right)$

And: $\frac{1}{2}\sinh 2u = \sinh u \cosh u = \sinh u\sqrt{1+\sinh^2 u} = x\sqrt{1+x^2}$

$$\therefore \quad \int \frac{x^2}{\sqrt{1+x^2}}\,dx = \underline{\underline{\frac{1}{2}\left[x\sqrt{1+x^2} - \ln\left(x + \sqrt{1+x^2}\right)\right] + k}}$$