P5 – COORDINATE SYSTEMS

Question:

The curve C has equation $y = 3\cosh\left(\frac{x}{3}\right)$.

(a) Show that the radius of curvature, at the point on C where $x = t$, is $3\cosh^2\left(\frac{t}{3}\right)$.

(b) Find the radius of curvature at the point where $t = 1.5$, giving your answer to 3 s.f.

(c) Find the area of the surface generated when the arc of C between $x = -3$ and $x = 3$ is rotated through 2π radians about the x-axis, giving your answer in terms of e and π.

[*Edexcel*]

Solution:

(a) $$\rho = \frac{\left[1 + \left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2\right]^{\frac{3}{2}}}{\frac{\mathrm{d}^2 y}{\mathrm{d}x^2}}$$

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \sinh\left(\frac{x}{3}\right) \quad \Rightarrow \quad \frac{\mathrm{d}^2 y}{\mathrm{d}x^2} = \tfrac{1}{3}\cosh\left(\frac{x}{3}\right)$$

$$\Rightarrow \quad \rho = \frac{\left[1 + \sinh^2\left(\frac{x}{3}\right)\right]^{\frac{3}{2}}}{\tfrac{1}{3}\cosh\left(\frac{x}{3}\right)} = \frac{3\left[\cosh^2\left(\frac{x}{3}\right)\right]^{\frac{3}{2}}}{\cosh\left(\frac{x}{3}\right)} = 3\cosh^2\left(\frac{x}{3}\right)$$

$$\therefore \quad \text{When } x = t,\ \rho = 3\cosh^2\left(\frac{t}{3}\right)$$

(b) Just put $t = 1.5$ into our expression and we get $\rho = 3.81$ (3 s.f.)

(c) $$\text{Surface area} = 2\pi\int_{-3}^{3} y\sqrt{1 + \left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2}\,\mathrm{d}x$$

$$= 2\pi\int_{-3}^{3} 3\cosh\left(\frac{x}{3}\right)\sqrt{1 + \sinh^2\left(\frac{x}{3}\right)}\,\mathrm{d}x = 2\pi\int_{-3}^{3} 3\cosh\left(\frac{x}{3}\right)\sqrt{\cosh^2\left(\frac{x}{3}\right)}\,\mathrm{d}x$$

$$= 2\pi\int_{-3}^{3} 3\cosh\left(\frac{x}{3}\right)\cdot\cosh\left(\frac{x}{3}\right)\mathrm{d}x = 6\pi\int_{-3}^{3}\cosh^2\left(\frac{x}{3}\right)\mathrm{d}x$$

$$= 3\pi\int_{-3}^{3}\left[\cosh\left(\frac{2x}{3}\right) + 1\right]\mathrm{d}x = 3\pi\left[\tfrac{3}{2}\sinh\left(\frac{2x}{3}\right) + x\right]_{-3}^{3}$$

$$= 3\pi\left[\tfrac{3}{2}\sinh(2) + 3 - \tfrac{3}{2}\sinh(-2) + 3\right] = 9\pi(\sinh 2 + 2) = 9\pi\left(\frac{e^2 - e^{-2}}{2} + 2\right)$$

$$\therefore \quad \text{Surface area} = \frac{9\pi}{2}\left(e^2 - e^{-2} + 4\right)$$