P5 – COORDINATE SYSTEMS

Question:

An ellipse has equation $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, where *a* and *b* are positive constants and *a* > *b*.

(a) Find an equation of the tangent at the point *P*(*a* cos *t*, *b* sin *t*).

(b) Find an equation of the normal at the point *P*(*a* cos *t*, *b* sin *t*).

The normal at *P* meets the *x*-axis at the point *Q*. The tangent at *P* meets the *y*-axis at the point *R*.

(c) Find, in terms of *a*, *b* and *t*, the coordinates of *M*, the mid-point of *QR*.

Given that $0<t<\frac{\pi}{2}$,

(d) Show that, as *t* varies, the locus of *M* has equation $\left(\frac{2ax}{a^2-b^2}\right)^2+\left(\frac{b}{2y}\right)^2=1$.

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Solution:

(a) $\frac{x}{a}\cos t+\frac{y}{b}\sin t=1$

(b) $ax\sin t-by\cos t=\left(a^2-b^2\right)\cos t\sin t$

(c) At *Q*, *y* = 0 so: $ax\sin t=\left(a^2-b^2\right)\cos t\sin t \quad\Rightarrow\quad ax=\left(a^2-b^2\right)\cos t$

$$\Rightarrow\quad x=\left(a-\frac{b^2}{a}\right)\cos t$$

At *R*, *x* = 0 so: $\frac{y}{b}\sin t=1 \quad\Rightarrow\quad y=\frac{b}{\sin t}=b\,\mathrm{cosec}\,t$

So *M* has coordinates: $\left(\tfrac{1}{2}\cos t\left(a-\frac{b^2}{a}\right),\ \tfrac{1}{2}b\,\mathrm{cosec}\,t\right)$.

(d) $x=\tfrac{1}{2}\left(a-\frac{b^2}{a}\right)\cos t=\tfrac{1}{2}\cos t\left(\frac{a^2-b^2}{a}\right) \quad\Rightarrow\quad 2ax=\left(a^2-b^2\right)\cos t \quad\Rightarrow\quad \cos t=\frac{2ax}{a^2-b^2}$

$y=\tfrac{1}{2}b\,\mathrm{cosec}\,t=\frac{b}{2\sin t} \quad\Rightarrow\quad \frac{2\sin t}{b}=\frac{1}{y} \quad\Rightarrow\quad \sin t=\frac{b}{2y}$

But: $\cos^2 t+\sin^2 t=1$

We can therefore write: $\left(\frac{2ax}{a^2-b^2}\right)^2+\left(\frac{b}{2y}\right)^2=1$