P5 – COORDINATE SYSTEMS

Question:

The rectangular hyperbola C has equation $xy = c^2$, where c is a positive constant.

(a) Show that the tangent to C at the point $P\left(cp, \frac{c}{p}\right)$ has equation $p^2 y = -x + 2cp$.

The point Q has coordinates $Q\left(cq, \frac{c}{q}\right)$, $q \neq p$. The tangents to C at P and Q meet at N.

Given that $p + q \neq 0$,

(b) Show that the y-coordinate of N is $\frac{2c}{p+q}$.

The line joining N to the origin O is perpendicular to the chord PQ.

(c) Find the numerical value of $p^2 q^2$.

[*Edexcel*]

Solution:

(a) $1 \cdot y + x \cdot \frac{\mathrm{d}y}{\mathrm{d}x} = 0 \quad \Rightarrow \quad \frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{y}{x}$

At P, $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{-\frac{c}{p}}{cp} = -\frac{1}{p^2}$

Therefore, equation of tangent:
$$y - \frac{c}{p} = -\frac{1}{p^2}(x - cp)$$
$$p^2 y - cp = -x + cp$$
$$\underline{\underline{p^2 y = -x + 2cp}}$$

(b) At N, $p^2 y = -x + 2cp$ and $q^2 y = -x + 2cq$.

Subtracting: $\left(p^2 - q^2\right) y = 2c(p - q)$
$$y = \frac{2c(p-q)}{p^2 - q^2} = \frac{2c(p-q)}{(p+q)(p-q)}$$
$$\underline{\underline{y = \frac{2c}{p+q}}}$$

(c) Note the statement preceding the question. We need the x-coordinate too. Multiplying the two expressions at the start of (b) by q^2 and p^2 respectively gives:

$$p^2 q^2 y = -q^2 x + 2cpq^2$$
$$p^2 q^2 y = -p^2 x + 2cp^2 q$$

Therefore: $-q^2 x + 2cpq^2 = -p^2 x + 2cp^2 q \quad \Rightarrow \quad \left(p^2 - q^2\right) x = 2cpq(p - q)$
$$x = \frac{2cpq(p-q)}{p^2 - q^2} = \frac{2cpq(p-q)}{(p+q)(p-q)} \quad \Rightarrow \quad x = \frac{2cpq}{p+q}$$

Now we find the gradients:

$$m_{ON} = \frac{\frac{2c}{p+q} - 0}{\frac{2cpq}{p+q} - 0} = \frac{2c}{2cpq} = \frac{1}{pq}$$

$$m_{PQ} = \frac{\frac{c}{q} - \frac{c}{p}}{cq - cp} = \frac{cp - cq}{cpq(q-p)} = \frac{p-q}{-pq(p-q)} = -\frac{1}{pq}$$

Since they are perpendicular: $m_{ON} \times m_{PQ} = -1$

$$\Rightarrow \quad \frac{1}{pq} \times -\frac{1}{pq} = -1$$

$$\Rightarrow \quad -\frac{1}{p^2q^2} = -1$$

Therefore: $\underline{\underline{p^2q^2 = 1}}$